

NASH EQUILIBRIUM WITH SUGENO PAYOFF

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ABSTRACT. This paper is devoted to Nash equilibrium for games in capacities. Such games with payoff expressed by Choquet integral were considered in [8] and existence of Nash equilibrium was proved. We also consider games in capacities but with expected payoff expressed by Sugeno integral. We prove existence of Nash equilibrium using categorical methods and abstract convexity theory.

1. INTRODUCTION

The classical Nash equilibrium theory is based on fixed point theory and was developed in frames of linear convexity. The mixed strategies of a player are probability (additive) measures on a set of pure strategies. But an interest to Nash equilibria in more general frames is rapidly growing in last decades. There are also results about Nash equilibrium for non-linear convexities. For instance, Briec and Horvath proved in [1] existence of Nash equilibrium point for B -convexity and MaxPlus convexity. Let us remark that MaxPlus convexity is related to idempotent (Maslov) measures in the same sense as linear convexity is related to probability measures.

We can use additive measures only when we know precisely probabilities of all events considered in a game. However it is not the case in many modern economic models. The decision theory under uncertainty considers a model when probabilities of states are either not known or imprecisely specified. Gilboa [7] and Schmeidler [18] axiomatized expectations expressed by Choquet integrals attached to non-additive measures called capacities, as a formal approach to decision-making under uncertainty. Dow and Werlang [3] generalized this approach for two players game where belief of each player about a choice of the strategy by the other player is a capacity. This result was extended onto games with arbitrary finite number of players [6].

Kozhan and Zaricnyi introduced in [8] a formal mathematical generalization of Dow and Werlang's concept of Nash equilibrium of a game where players are allowed to form non-additive beliefs about opponent's decision but also to play their mixed non-additive strategies. Such game is called by authors game in capacities. The expected payoff function was there defined using a Choquet integral. Kozhan and Zaricnyi proved existence theorem using a linear convexity on the space of capacities which is preserved by Choquet integral. There was stated a problem of existence of Nash equilibrium for another functors [8].

An alternative to so-called Choquet expected utility model is the qualitative decision theory. The corresponding expected utility is expressed by Sugeno integral.

See for example papers [4], [5], [2], [17] and others. Sugeno integral chooses a median value of utilities which is qualitative counterpart of the averaging operation by Choquet integral.

Following [8] we introduce in this paper the general mathematical concept of Nash equilibrium of a game in capacities. However, motivated by the qualitative approach, we consider expected payoff function defined by Sugeno integral. To prove existence theorem for this concrete case, we consider more general framework which could unify all mentioned before situations and give us a method to prove theorems about existence of Nash equilibrium in different contexts. We use categorical methods and abstract convexity theory.

The notion of convexity considered in this paper is considerably broader than the classic one; specifically, it is not restricted to the context of linear spaces. Such convexities appeared in the process of studying different structures like partially ordered sets, semilattices, lattices, superextensions etc. We base our approach on the notion of topological convexity from [21] where the general convexity theory is covered from axioms to application in different areas. Particularly, there is proved Kakutani fixed point theorem for abstract convexity.

Above mentioned constructions of the spaces of probability measures, idempotent measures and capacities are functorial and could be completed to monads (see [16], [23] and [12] for more details). There was introduced in [13] a convexity structure on each \mathbb{F} -algebra for any monad \mathbb{F} in the category of compact Hausdorff spaces and continuous maps. Particularly, topological properties of monads with binary convexities were investigated.

We prove a counterpart of Nash theorem for an abstract convexity in this paper. Particularly, we consider binary convexities. These results we use to obtain Nash theorem for algebras of any L-monad with binary convexity. Since capacity monad is an L-monad with binary convexity [14], we obtain as corollary the corresponding result for capacities.

2. GAMES IN CAPACITIES

By **Comp** we denote the category of compact Hausdorff spaces (compacta) and continuous maps. For each compactum X we denote by $C(X)$ the Banach space of all continuous functions on X with the usual sup-norm. In what follows, all spaces and maps are assumed to be in **Comp** except for \mathbb{R} and maps in sets $C(X)$ with X compact Hausdorff.

We need the definition of capacity on a compactum X . We follow a terminology of [12]. A function c which assigns each closed subset A of X a real number $c(A) \in [0, 1]$ is called an *upper-semicontinuous capacity* on X if the three following properties hold for each closed subsets F and G of X :

1. $c(X) = 1$, $c(\emptyset) = 0$,
2. if $F \subset G$, then $c(F) \leq c(G)$,
3. if $c(F) < a$, then there exists an open set $O \supset F$ such that $c(B) < a$ for each compactum $B \subset O$.

We extend a capacity c to all open subsets $U \subset X$ by the formula $c(U) = \sup\{c(K) \mid K \text{ is a closed subset of } X \text{ such that } K \subset U\}$.

It was proved in [12] that the space MX of all upper-semicontinuous capacities on a compactum X is a compactum as well, if a topology on MX is defined by a subbase that consists of all sets of the form $O_-(F, a) = \{c \in MX \mid c(F) < a\}$, where F is a closed subset of X , $a \in [0, 1]$, and $O_+(U, a) = \{c \in MX \mid c(U) > a\}$, where U is an open subset of X , $a \in [0, 1]$. Since all capacities we consider here are upper-semicontinuous, in the following we call elements of MX simply capacities.

There is considered in [8] a tensor product for capacities, which is a continuous map $\otimes : MX_1 \times \cdots \times MX_n \rightarrow M(X_1 \times \cdots \times X_n)$. Note that, despite the space of capacities contains the space of probability measures, the tensor product of capacities does not extend tensor product of probability measures.

Due to Zhou [24] we can identify the set MX with some set of functionals defined on the space $C(X)$ using the Choquet integral. We consider for each $\mu \in MX$ its value on a function $f \in C(X)$ defined by the formulae

$$\mu(f) = \int f d\mu = \int_0^\infty \mu\{x \in X | f(X) \geq t\} dt + \int_{-\infty}^0 (\mu\{x \in X | f(X) \geq t\} - 1) dt$$

Let us remember the definition of Nash equilibrium. We consider a n -players game $f : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ with compact Hausdorff spaces of strategies X_i . The coordinate function $f_i : X \rightarrow \mathbb{R}$ we call payoff function of i -th player. For $x \in X$ and $t_i \in X_i$ we use the notation $(x; t_i) = (x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)$. A point $x \in X$ is called a Nash equilibrium point if for each $i \in \{1, \dots, n\}$ and for each $t_i \in X_i$ we have $f_i(x; t_i) \leq f_i(x)$. Kozhan and Zarichnyj proved in [8] existence of Nash equilibrium for game in capacities $ef : \prod_{i=1}^n MX_i \rightarrow \mathbb{R}^n$ with expected payoff functions defined by

$$ef_i(\mu_1, \dots, \mu_n) = \int_{X_1 \times \cdots \times X_n} f_i d(\mu_1 \otimes \cdots \otimes \mu_n)$$

Let us remark that the Choquet functional representation of capacities preserves the natural linear convexity structure on MX which was used in the proof of existence of Nash equilibrium [8]. However this representation does not preserve the capacity monad structure. (We will introduce the monad notion in Section 4).

There was introduced [14] another functional representation of capacities using Sugeno integral (see also [11] for similar result). This representation preserves the capacity monad structure. Let us describe such representation. Fix any increasing homeomorphism $\psi : (0, 1) \rightarrow \mathbb{R}$. We put additionally $\psi(0) = -\infty$, $\psi(1) = +\infty$ and assume $-\infty < t < +\infty$ for each $t \in \mathbb{R}$. We consider for each $\mu \in MX$ its value on a function $f \in C(X)$ defined by the formulae

$$\mu(f) = \int_X^{Sug} f d\mu = \max\{t \in \mathbb{R} \mid \mu(f^{-1}([t, +\infty))) \geq \psi^{-1}(t)\}$$

Let us remark that we use some modification of Sugeno integral. The original Sugeno integral [19] "ignores" function values outside the interval $[0, 1]$ and we introduce a "correction" homeomorphism ψ to avoid this problem. Now, following [8], we consider a game in capacities $sf : \prod_{i=1}^n MX_i \rightarrow \mathbb{R}^n$, but motivated by [4], we consider Sugeno expected payoff functions defined by

$$sf_i(\mu_1, \dots, \mu_n) = \int_{X_1 \times \cdots \times X_n}^{Sug} f_i d(\mu_1 \otimes \cdots \otimes \mu_n)$$

The main goal of this paper is to prove existence of Nash equilibrium for such game. Since Sugeno integral does not preserve linear convexity on MX we can not use methods from [8]. We will use some another natural convexity structure which has the binarity property (has Helly number 2). We will obtain some general result for such convexities which could be useful to investigate existence of Nash equilibrium for diverse construction. Finally, we will obtain the result for capacities as a corollary of these general results.

3. BINARY CONVEXITIES

A family \mathcal{C} of closed subsets of a compactum X is called a *convexity* on X if \mathcal{C} is stable for intersection and contains X and the empty set. Elements of \mathcal{C} are

called \mathcal{C} -convex (or simply convex). Although we follow general concept of abstract convexity from [21], our definition is different. We consider only closed convex sets. Such structure is called closure structure in [21]. The whole family of convex sets in the sense of [21] could be obtained by the operation of union of up-directed families. In what follows, we assume that each convexity contains all singletons.

A convexity \mathcal{C} on X is called T_2 if for each distinct $x_1, x_2 \in X$ there exist $S_1, S_2 \in \mathcal{C}$ such that $S_1 \cup S_2 = X$, $x_1 \notin S_2$ and $x_2 \notin S_1$. Let us remark that if a convexity \mathcal{C} on a compactum X is T_2 , then \mathcal{C} is a subbase for closed sets. A convexity \mathcal{C} on X is called T_4 (normal) if for each disjoint $C_1, C_2 \in \mathcal{C}$ there exist $S_1, S_2 \in \mathcal{C}$ such that $S_1 \cup S_2 = X$, $C_1 \cap S_2 = \emptyset$ and $C_2 \cap S_1 = \emptyset$.

Let $(X, \mathcal{C}), (Y, \mathcal{D})$ be two compacta with convexity structures. A continuous map $f : X \rightarrow Y$ is called *CP-map* (convexity preserving map) if $f^{-1}(D) \in \mathcal{C}$ for each $D \in \mathcal{D}$; f is called *CC-map* (convex-to-convex map) if $f(C) \in \mathcal{D}$ for each $C \in \mathcal{C}$.

By a multimap (set-valued map) of a set X into a set Y we mean a map $F : X \rightarrow 2^Y$. We use the notation $F : X \multimap Y$. If X and Y are topological spaces, then a multimap $F : X \multimap Y$ is called upper semi-continuous (USC) provided for each open set $O \subset Y$ the set $\{x \in X \mid F(x) \subset O\}$ is open in X . It is well-known that a multimap is USC iff its graph is closed in $X \times Y$.

Let $F : X \multimap X$ be a multimap. We say that a point $x \in X$ is a fixed point of F if $x \in F(x)$. The following counterpart of Kakutani theorem for abstract convexity is a partial case of Theorem 3 from [22] (it also could be obtain combining Theorem 6.15, Ch.IV and Theorem 4.10, Ch.III from [21]).

Theorem 1. *Let \mathcal{C} be a normal convexity on a compactum X such that all convex sets are connected and $F : X \multimap X$ is a USC multimap with values in \mathcal{C} . Then F has a fixed point.*

Let \mathcal{C} be a family of subsets of a compactum X . We say that \mathcal{C} is *linked* if the intersection of every two elements is non-empty. A convexity \mathcal{C} is called *binary* if the intersection of every linked subsystem of \mathcal{C} is non-empty.

Lemma 1. *Let \mathcal{C} be a T_2 binary convexity on a continuum X . Then \mathcal{C} is normal and all convex sets are connected.*

Proof. The first assertion of the lemma is proved in Lemma 3.1 [16]. Let us prove the second one. Consider any $A \in \mathcal{C}$. There was defined in [10] a retraction $h_A : X \rightarrow A$ by the formula $h_A(x) = \cap \{C \in \mathcal{C} \mid x \in C \text{ and } C \cap A \neq \emptyset\}$. Hence A is connected and the lemma is proved. \square

Now we can reformulate Theorem 1 for binary convexities.

Theorem 2. *Let \mathcal{C} be a T_2 binary convexity on a continuum X and $F : X \multimap X$ is a USC multimap with values in \mathcal{C} . Then F has a fixed point.*

Now, let \mathcal{C}_i be a convexity on X_i . We say that the function $f_i : X \rightarrow \mathbb{R}$ is quasi concave by i -th coordinate if we have $(f_i^x)^{-1}([t; +\infty)) \in \mathcal{C}_i$ for each $t \in \mathbb{R}$ and $x \in X$ where $f_i^x : X_i \rightarrow \mathbb{R}$ is a function defined as follows $f_i^x(t_i) = f_i(x; t_i)$ for $t_i \in X_i$.

Theorem 3. *Let $f : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ be a game with a normal convexity \mathcal{C}_i defined on each compactum X_i such that all convex sets are connected, the function f is continuous and the function $f_i : X \rightarrow \mathbb{R}$ is quasi concave by i -th coordinate for each $i \in \{1, \dots, n\}$. Then there exists a Nash equilibrium point.*

Proof. Fix any $x \in X$. For each $i \in \{1, \dots, n\}$ consider a set $M_i^x \subset X_i$ defined as follows $M_i^x = \{t \in X_i \mid f_i^x(t) = \max_{s \in X_i} f_i^x(s)\}$. We have that M_i^x is a closed subset X_i . Since the function $f_i : X \rightarrow \mathbb{R}$ is quasi concave by i -th coordinate,

we have that $M_i^x \in \mathcal{C}_i$. Define a multimap $F : X \multimap X$ by the formulae $F(x) = \prod_{i=1}^n M_i^x$ for $x \in X$.

Let us show that F is USC. Consider any point $(x, y) \in X \times X$ such that $y \notin F(x)$. Then there exists $i \in \{1, \dots, n\}$ such that $f_i^x(y_i) < \max_{s \in X_i} f_i^x(s)$. Hence we can choose $t_i \in X_i$ such that $f_i(x; y_i) < f_i(x; t_i)$. Since f_i is continuous, there exists a neighborhood O_x of x in X and a neighborhood O_{y_i} of y_i in Y_i such that for each $x' \in O_x$ and $y'_i \in O_{y_i}$ we have $f_i(x; y'_i) < f_i(x; t_i)$. Put $O_y = (\text{pr}_i)^{-1}(O_{y_i})$. Then for each $(x', y') \in O_x \times O_y$ we have $y' \notin F(x')$. Thus the graph of F is closed in $X \times Y$, hence F is upper semicontinuous.

We consider on X the family $\mathcal{C} = \{\prod_{i=1}^n C_i \mid C_i \in \mathcal{C}_i\}$. It is easy to see that \mathcal{C} forms a normal convexity on compactum X such that all convex sets are connected. Then by Theorem 1 F has a fixed point which is a Nash equilibrium point. \square

Now, the following corollary follows from the previous theorem and Lemma 1.

Corollary 1. *Let $f : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ be a game such that there is defined a T_2 binary convexity \mathcal{C}_i on each continuum X_i , the function f is continuous and the function $f_i : X \rightarrow \mathbb{R}$ is quasi concave by i -th coordinate for each $i \in \{1, \dots, n\}$. Then there exists a Nash equilibrium point.*

4. L-MONADS AND ITS ALGEBRAS

We apply Corollary 1 to study games defined on algebras of binary L-monads. We recall some categorical notions (see [9] and [20] for more details). We define them only for the category **Comp**. Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a covariant functor. A functor F is called continuous if it preserves the limits of inverse systems. In what follows, all functors assumed to preserve monomorphisms, epimorphisms, weight of infinite compacta. We also assume that our functors are continuous. For a functor F which preserves monomorphisms and an embedding $i : A \rightarrow X$ we shall identify the space FA and the subspace $F(i)(FA) \subset FX$.

A monad $\mathbb{T} = (T, \eta, \mu)$ in the category **Comp** consists of an endofunctor $T : \mathbf{Comp} \rightarrow \mathbf{Comp}$ and natural transformations $\eta : \text{Id}_{\mathbf{Comp}} \rightarrow T$ (unity), $\mu : T^2 \rightarrow T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \text{id}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By $\text{Id}_{\mathbf{Comp}}$ we denote the identity functor on the category **Comp** and T^2 is the superposition $T \circ T$ of T .)

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category **Comp**. The pair (X, ξ) where $\xi : TX \rightarrow X$ is a map is called a \mathbb{T} -algebra if $\xi \circ \eta X = \text{id}_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let $(X, \xi), (Y, \xi')$ be two \mathbb{T} -algebras. A map $f : X \rightarrow Y$ is called a \mathbb{T} -algebras morphism if $\xi' \circ Tf = f \circ \xi$.

Let (X, ξ) be an \mathbb{F} -algebra for a monad $\mathbb{F} = (F, \eta, \mu)$ and A is a closed subset of X . Denote by f_A the quotient map $f_A : X \rightarrow X/A$ (the classes of equivalence are one-point sets $\{x\}$ for $x \in X \setminus A$ and the set A) and put $a = f_A(A)$. Denote $A^+ = (Ff_A)^{-1}(\eta(X/A)(a))$. Define the \mathbb{F} -convex hull $C_{\mathbb{F}}(A)$ of A as follows $C_{\mathbb{F}}(A) = \xi(A^+)$. Put additionally $C_{\mathbb{F}}(\emptyset) = \emptyset$. We define the family $\mathcal{C}_{\mathbb{F}}(X, \xi) = \{A \subset X \mid A \text{ is closed and } C_{\mathbb{F}}(A) = A\}$. Elements of the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ we call \mathbb{F} -convex. It was shown in [13] that the family $\mathcal{C}_{\mathbb{F}}(X, \xi)$ forms a convexity on X , moreover, each morphism of \mathbb{F} -algebras is a CP -map. Let us remark that one-point sets are always \mathbb{F} -convex.

We don't know if the convexities we have introduced are T_2 . We consider in this section a class of monads generating convexities which have this property. The class of L -monads was introduced in [13] and it contains many well-known monads in **Comp** like superextension, hyperspace, probability measure, capacity, idempotent measure etc. For $\phi \in C(X)$ by $\max \phi$ ($\min \phi$) we denote $\max_{x \in X} \phi(x)$ ($\min_{x \in X} \phi(x)$) and π_ϕ or $\pi(\phi)$ denote the corresponding projection

$\pi_\phi : \prod_{\psi \in C(X)} [\min \psi, \max \psi] \rightarrow [\min \phi, \max \phi]$. It was shown in [15] that for each L-monad $\mathbb{F} = (F, \eta, \mu)$ we can consider FX as subset of the product $\prod_{\phi \in C(X)} [\min \phi, \max \phi]$, moreover, we have $\pi_\phi \circ \eta X = \phi$, $\pi_\phi \circ \mu X = \pi(\pi_\phi)$ for all $\phi \in C(X)$ and $\pi_\psi \circ Ff = \pi_{\psi \circ f}$ for all $\psi \in C(Y)$, $f : X \rightarrow Y$. We could consider these properties of L-monads as a definition [15].

We say that an L-monad $\mathbb{F} = (F, \eta, \mu)$ weakly preserves preimages if for each map $f : X \rightarrow Y$ and each closed subset $A \subset Y$ we have $\pi_\phi(\nu) \in [\min \phi(f^{-1}(A)), \max \phi(f^{-1}(A))]$ for each $\nu \in (Ff)^{-1}(A)$ and $\phi \in C(X)$ [13]. It was shown in [13] that for each L-monad \mathbb{F} which weakly preserves preimages the convexity $\mathcal{C}_{\mathbb{F}}(FX, \mu X)$ is T_2 .

Lemma 2. *Let (X, ξ) be an \mathbb{F} -algebra for an L-monad $\mathbb{F} = (F, \eta, \mu)$ which weakly preserves preimages. Then the map $\xi : FX \rightarrow X$ is a CC-map for convexities $\mathcal{C}_{\mathbb{F}}(FX, \mu)$ and $\mathcal{C}_{\mathbb{F}}(X, \xi)$ respectively.*

Proof. Consider any $B \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$. We should show that $\xi(B) \in \mathcal{C}_{\mathbb{F}}(X, \xi)$. Denote by $\chi : X \rightarrow X/\xi(B)$ the quotient map and put $b = \chi(\xi(B))$. Consider any $\mathcal{A} \in FX$ such that $F\chi(\mathcal{A}) = (\eta(X/\xi(B)))(b)$. We should show that $\xi(\mathcal{A}) \in \xi(B)$.

Consider the quotient map $\chi_1 : FX \rightarrow FX/B$ and put $b_1 = \chi_1(B)$. There exists a (unique) continuous map $\xi' : FX/B \rightarrow X/\xi(B)$ such that $\xi'(b_1) = b$ and $\xi' \circ \chi_1 = \chi \circ \xi$. Put $\mathcal{D} = F(\eta X)(\mathcal{A})$. We have $F\xi(\mathcal{D}) = \mathcal{A}$, hence $F\xi' \circ F\chi_1(\mathcal{D}) = F\chi \circ F\xi(\mathcal{D}) = F\chi(\mathcal{A}) = \eta(X/\xi(B))(b)$. Since F weakly preserves preimages, we have $F\chi_1(\mathcal{D}) = \eta(FX/B)(b_1)$. Since $B \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$, we have $\mu X(\mathcal{D}) \in B$. Hence $\xi(\mathcal{A}) = \xi \circ F\xi(\mathcal{D}) = \xi \circ \mu(\mathcal{D}) \in \xi(B)$. The lemma is proved. \square

We call a monad \mathbb{F} binary if $\mathcal{C}_{\mathbb{F}}(X, \xi)$ is binary for each \mathbb{F} -algebra (X, ξ) .

Lemma 3. *Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. Then for each \mathbb{F} -algebra (X, ξ) the convexity $\mathcal{C}_{\mathbb{F}}(X, \xi)$ is T_2 .*

Proof. Consider any two distinct points $x, y \in X$. Since ξ is a morphism of \mathbb{F} -algebras $(FX, \mu X)$ and (X, ξ) , it is a CP-map and we have $\xi^{-1}(x), \xi^{-1}(y) \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$. Since $\mathcal{C}_{\mathbb{F}}(FX, \mu)$ is T_2 and binary, it is normal by Lemma 1. Hence we can choose $L_1, L_2 \in \mathcal{C}_{\mathbb{F}}(FX, \mu)$ such that $L_1 \cup L_2 = FX$ and $L_1 \cap \xi^{-1}(x) = \emptyset$, $L_2 \cap \xi^{-1}(y) = \emptyset$. Then we have $\xi(L_1), \xi(L_2) \in \mathcal{C}_{\mathbb{F}}(X, \xi)$ by Lemma 2, $\xi(L_1) \cup \xi(L_2) = X$, $x \notin \xi(L_1)$ and $y \notin \xi(L_2)$. The lemma is proved. \square

Consider any L-monad $\mathbb{F} = (F, \eta, \mu)$. It is easy to check that for each segment $[a, b] \subset \mathbb{R}$ the pair $([a, b], \xi_{[a, b]})$ is an F -algebra where $\xi_{[a, b]} = \pi_{\text{id}_{[a, b]}}$. Consider a game $f : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ where for each compactum X_i there exists a map $\xi_i : FX_i \rightarrow X_i$ such that the pair (X_i, ξ_i) is an \mathbb{F} -algebra. We say that the function $f_i : X \rightarrow \mathbb{R}$ is an \mathbb{F} -algebras morphism by i -th coordinate if for each $x \in X$ the function $f_i^x : X_i \rightarrow \mathbb{R}$ is a morphism of \mathbb{F} -algebras (X_i, ξ_i) and $([\min f_i^x, \max f_i^x], \xi_{[\min f_i^x, \max f_i^x]})$.

Theorem 4. *Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. Let $f : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ be a game such that there is defined an \mathbb{F} -algebra map $\xi_i : FX_i \rightarrow X_i$ on each continuum X_i , the function f is continuous and the function $f_i : X \rightarrow \mathbb{R}$ is an \mathbb{F} -algebras morphism by i -th coordinate for each $i \in \{1, \dots, n\}$. Then there exists a Nash equilibrium point.*

Proof. Since for each $x \in X$ the function $f_i^x : X_i \rightarrow \mathbb{R}$ is an \mathbb{F} -algebras morphism, it is a CP-map, hence quasi concave. Now, our theorem follows from Lemma 3 and Corollary 1. \square

5. PURE AND MIXED STRATEGIES

Let $\mathbb{F} = (F, \eta, \mu)$ be a binary L-monad which weakly preserves preimages. We consider Nash equilibrium for free algebras $(FX, \mu X)$ in this section. Points of a compactum X we call pure strategies and points of FX we call mixed strategies. Such approach is a natural generalization of the model from [8] where spaces of capacities MX were considered.

We consider a game $u : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}^n$ with compact Hausdorff spaces of pure strategies X_1, \dots, X_n and continuous payoff functions $u_i : \prod_{i=1}^n X_i \rightarrow \mathbb{R}$.

It is well known how to construct the tensor product of two (or finite number) probability measures. This operation was generalized in [20] for each monad in the category **Comp**. More precisely there was constructed for each compacta X_1, \dots, X_n a continuous map $\otimes : \prod_{i=1}^n FX_i \rightarrow F(\prod_{i=1}^n X_i)$ which is natural by each argument and for each i we have $F(p_i) \circ \otimes = \text{pr}_i$ where $p_i : \prod_{j=1}^n X_j \rightarrow X_i$ and $\text{pr}_i : \prod_{j=1}^n FX_j \rightarrow FX_i$ are natural projections.

We define the payoff functions $eu_i : FX_1 \times \dots \times FX_n \rightarrow \mathbb{R}$ by the formula $eu_i = \pi_{u_i} \circ \otimes$. Evidently, eu_i is continuous. Consider any $t \in \mathbb{R}$ and $\nu \in FX_1 \times \dots \times FX_n$. Then we have $(eu_i^\nu)^{-1}[t; +\infty) = \{\mu_i \in FX_i \mid eu_i(\nu; \mu_i) \geq t_i\} = l^{-1}(\pi_{u_i}^{-1}[t; +\infty) \cap \{\nu_i\} \times \dots \times FX_i \times \dots \times \{\nu_n\})$, where $l : FX_i \rightarrow \prod_{j=1}^n FX_j$ is an embedding defined by $l(\mu_i) = (\nu; \mu_i)$ for $\mu_i \in FX_i$. A structure of \mathbb{F} -algebra on the product $\prod_{j=1}^n FX_j$ of \mathbb{F} -algebras $(FX_i, \mu X_i)$ is given by a map $\xi : F(\prod_{i=1}^n FX_i) \rightarrow \prod_{i=1}^n FX_i$ defined by the formula $\xi = (\mu X_i \circ F(p_i))_{i=1}^n$. It is easy to check that a product of convex in FX_i sets is convex in $\prod_{i=1}^n FX_i$. Since \mathbb{F} weakly preserves preimages, $\pi_{u_i}^{-1}[t; +\infty)$ is convex in $\prod_{i=1}^n FX_i$. It is easy to see that l is a CP-map, hence the map eu_i is quasiconcave on i -th coordinate.

Hence, using Corollary 1, we obtain the following theorem.

Theorem 5. *The game with payoff functions eu_i has a Nash equilibrium point provided each FX_i is connected.*

Now, consider a game in capacities with Sugeno payoff functions introduced in the beginning of the paper.

The assignment M extends to the capacity functor M in the category of compacta, if the map $Mf : MX \rightarrow MY$ for a continuous map of compacta $f : X \rightarrow Y$ is defined by the formula $Mf(c)(F) = c(f^{-1}(F))$ where $c \in MX$ and F is a closed subset of X . This functor was completed to the monad $\mathbb{M} = (M, \eta, \mu)$ [12], where the components of the natural transformations are defined as follows: $\eta X(x)(F) = 1$ if $x \in F$ and $\eta X(x)(F) = 0$ if $x \notin F$; $\mu X(\mathcal{C})(F) = \sup\{t \in [0, 1] \mid \mathcal{C}(\{c \in MX \mid c(F) \geq t\}) \geq t\}$, where $x \in X$, F is a closed subset of X and $\mathcal{C} \in M^2(X)$. Since capacity monad \mathbb{M} is a binary L-monad which weakly preserves preimages with $\pi_\varphi(\nu) = \int_X^{Sug} f d\nu$ for any $\nu \in MX$ and $\varphi \in C(X)$ [14], we obtain as a consequence

Corollary 2. *A game in capacities $sf : \prod_{i=1}^n MX_i \rightarrow \mathbb{R}^n$ with Sugeno payoff functions has a Nash equilibrium point.*

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